Spinors Part 1: Bloch Sphere and the Hopf Fibration

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I. INTRODUCTION

In introductory quantum mechanics, spin-$\frac{1}{2}$ systems are often given as the first example of a system where classical physics can't capture the full story. The Stern-Gerlach experiment, which involves shooting silver atoms through a magnetic field and measuring their deflection, defies classical intuition as we observe only a discrete spectrum of magnetic moments (and hence angular momenta), as opposed to a continuous range of values. This lead to the development of the theory of quantised angular momentum and the description of an additional intrinsic quantum property, called spin. For the electrons in the silver atoms, which are spin-$\frac{1}{2}$ particles, they can take a binary spectrum: spin up and spin down.

When measured, the spin of particles like electrons behave probabilistically. The state of the electron before measurement is described by an expression called the *wave function*, and the particle's interaction with the magnetic field and subsequent measurement alters the wave function describing the particle. Due to the probabilistic nature of the wave function, we can only know the probabilities of the associated pure states (spin up or down). Measurement of a mutually orthogonal spin axis after the initial measurement erases the information of the original state, meaning the wave function describing the original spin axis returns to being a superposition of states rather than a pure state.

This is where most of us come across the Bloch sphere as a way to represent these qubit systems (qubit being short for quantum bit, since it can be in any superposition of the two pure states). The Bloch sphere is an informative visual tool that represents the space of states for our qubit's wave function and how it can evolve with time. Much of the mathematical underpinnings are left to students to study individually; in aid of this pursuit, we aim to explain more of the mathematical background that underlies this modest ball and the wider theory of spinors in a series of two articles, of which this is the first. (Part 2 will be published in Jeremy Issue 4, 2025.)

II. THE FULL STORY MADE SHORT

Here is our journey for Part 1 laid out from start to finish:

$$\begin{tikzcd} \mathbb{C}^2 \\ S^3 \arrow[u, hook] \arrow[r] & S^2 \cong \mathbb{CP}^1 \end{tikzcd}$$

where the hook-arrow denotes an *embedding* of one space into another. We will explain each step along this path and context/motivation as we go along.

III. THE SHORT STORY MADE LONG

Our qubit's wave function, denoted $|\psi\rangle$, can be influenced and interfered with by the system in ways reminiscent of physical waves. This motivates the use of complex numbers in the construction of our state space. We treat the two independent measurement outcomes (spin up and down) as two linearly independent vectors $|+\rangle$ and $|-\rangle$ respectively, and then allow these vectors to be scaled by complex coefficients α, β; thus, our state space is $\mathbb{C}^2$.

Motivated by the idea that measuring the "overlap" of states can predict the probability of transitioning from one state to another, we introduce an inner product on our finite-dimensional vector space, given by

$$\langle\psi|\varphi\rangle = \alpha_\psi^* \alpha_\varphi + \beta_\psi^* \beta_\varphi. \tag{1}$$

where $*$ denotes complex conjugation. This leads to the notion of a norm on our state space given by [1]

$$||\,|\psi\rangle\,|| = \sqrt{\langle\psi|\psi\rangle} = \sqrt{|\alpha|^2 + |\beta|^2}, \tag{2}$$

and to ensure our probabilities add up to 1 our state vectors are normalised, so that $|\alpha|^2 + |\beta|^2 = 1$. With these features (inner product and *completeness* (where Cauchy sequences converge) coming from the vector space being finite-dimensional), we get a *Hilbert space*, which is the typical type of space needed for representing a quantum system.

Normalisation of $\mathbb{C}^2$ vectors maps states $|\psi\rangle \mapsto \frac{|\psi\rangle}{|||\psi\rangle||}$, meaning states that differ only by a multiple of a real number are mapped to the same unit vector, called a *spinor*. These live on the unit 3-sphere

$$S^3 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\} \tag{3}$$

which is described by three real coordinates embedded in a two complex-dimensional (i.e. four real-dimensional) space. Hence we write $S^3 \hookrightarrow \mathbb{C}^2$. However, there is another form of redundancy in our states. Any two states that differ by a multiple of a global phase $e^{i\theta}$ are indistinguishable by measurements according to the Born rule, so they represent the same physical state despite being different spinors. Thus the physical state space can be thought of as the set of "equivalence classes" of spinors when we divide out a factor of $e^{i\theta}$. This is what we call the Bloch sphere.

[1] This is just the discrete L^2-norm.

IV. FIRST GROUP CONNECTIONS

We begin by outlining the relevant groups needed to discuss the formalism behind the Bloch sphere's construction.

The unitary group of degree one

$$\mathrm{U}(1) = \{z \in \mathbb{C} : |z| = 1\} \tag{4}$$

is the set of complex numbers with magnitude 1. These are of the form $e^{i\theta}$, where θ is some angle, and thus U(1) is just the unit circle in $\mathbb{C}$ and its topology is identified with the circle S^1.

The Special Unitary group of degree 2

$$\mathrm{SU}(2) = \{U \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) : U^\dagger U = I, \det U = 1\} \tag{5}$$

has elements of the form

$$U = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}. \tag{6}$$

The determinant condition is equivalent to

$$|\alpha|^2 + |\beta|^2 = 1, \tag{7}$$

which should ring some bells as this is the norm-squared of a spinor $|\psi\rangle = [\alpha, \beta] \in S^3 \hookrightarrow \mathbb{C}^2$. SU(2) is also described with 3 real components, like the spinors prior to removal of the global phase, so one can construct a *diffeomorphism* from $\mathrm{SU}(2) \to S^3$ that allows us to identify the group SU(2) with the topological structure of S^3 and vice versa:

$$[\alpha, \beta] \mapsto U = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \in \mathrm{SU}(2)$$

Thus, the 3-sphere $S^3 \cong \mathrm{SU}(2)$ as smooth manifolds, and we find that spinors and their corresponding SU(2) matrices encode the same information. The matrix contains the corresponding spinor as its first column and its orthonormal pair as the second column, satisfying the $\det U = 1$ condition. However, there is a difference between a SU(2) matrix and its corresponding spinor, and that difference is that the former can act on the latter to transform spin states; this difference, and its consequences, will be made clear in the mathematical formalism developed in Part 2 of this series.

V. SPINORS AND THE RIEMANN SPHERE

We're now going to slow down and consider spinors from first principles, along the way discussing how:

1. Spinors represent a body's rotation by encoding information about the axis and direction of rotation.

2. Spinors that are orthogonal represent antipodal states on the Bloch sphere, and vice versa. (This seems unintuitive because orthogonal vectors are usually thought of as being perpendicular, not opposite.)

3. Rotating the physical state space by an angle 2π returns the spinor $-|\psi\rangle$, and rotating again by 2π gets you back to $|\psi\rangle$. In general, rotating a physical state vector by a given angle only rotates the corresponding spinor by half that angle, and vice versa.

In fact, only the first two properties will be discussed here. The third will be thoroughly discussed in Part 2 of this series.

Spinors often seem unintuitive because there isn't an easy way to plot them or their transformations, since we need four dimensions to plot S^3 and visualise the effect of multiplying a spinor by an SU(2) matrix. Instead, we look at the construction of spinors and their transformations through the lens of the Riemann sphere. The Riemann sphere is the most basic object in complex analysis as it provides the codomain for general complex functions, and a both thorough and wonderfully approachable treatment of it can be found it [1]. We will show it coincides with the Bloch sphere through their shared equivalence to the complex projective line $\mathbb{CP}^1$, and construct the theory of spinors around it. In Part 2 we will see that the additional complex structure it wields allows for a better (and animatable) visualisation of spinor transformations.

A. The Riemann sphere is $\mathbb{CP}^1$ is the Bloch sphere

We start with the unit sphere S^2 embedded in $\mathbb{R}^3$ and centered on the origin, and overlay the complex plane $\mathbb{C}^1$ onto the xy plane such that the real axis overlaps the x-axis and the imaginary axis the y-axis. Then for any point (x, y, z) on S^2, its stereographic projection is the complex number ζ that sits at the intersection of the xy plane and the line connecting the South Pole to (x, y, z). (See Fig. 1.) Algebraically, this defines the map

$$(x, y, z) \mapsto \frac{x + iy}{1 + z}, \tag{8}$$

which has inverse

$$\zeta \mapsto \left(\frac{2\mathrm{Re}(\zeta)}{1 + \zeta\zeta^*}, \frac{2\mathrm{Im}(\zeta)}{1 + \zeta\zeta^*}, \frac{1 - \zeta\zeta^*}{1 + \zeta\zeta^*} \right). \tag{9}$$

The North Pole is mapped to 0, the equator $x^2 + y^2 = 1$ is mapped to the unit circle $S^1 \equiv \mathrm{U}(1)$, and the entire northern hemisphere is mapped to the interior of the unit disc. Similarly, the southern hemisphere is mapped to the exterior (i.e. complement) of the unit disc. However, as the point on the sphere approaches the South Pole, the projection line approaches horizontal and the modulus of the projected number grows without bound; hence we introduce a "number at infinity", called ∞, and set the projection of the South Pole to be this point. The resulting set

$$\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\} \tag{10}$$

is called the extended complex plane, and the stereographic projection is a homeomorphism between it and the sphere, thus allowing us to treat the two objects as topologically equivalent and justifying the name "Riemann sphere".

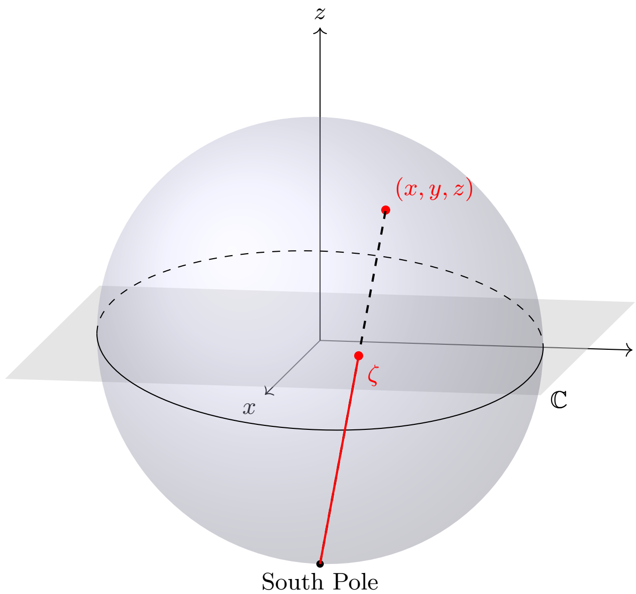

FIG. 1. Stereographic Projection of the point (x, y, z) on the unit sphere to $\zeta \in \mathbb{C}$.

The complex projective line $\mathbb{CP}^1$ is constructed by starting with $\mathbb{C}^2 \setminus [0, 0]$ and quotienting out the equivalence relation $[a, b] \sim \lambda[a, b] \quad \forall \lambda \in \mathbb{C} \setminus \{0\}$, so that all scalar multiples of vectors are made equivalent. Denoting the equivalence class of $[a, b]$ as $[a : b]$, the easiest choice of representative when $b \neq 0$ is $[a/b, 1]$, while for $b = 0$ there is only one equivalence class $[1 : 0]$. Thus the complex projective line is

$$\mathbb{CP}^1 = \{[\zeta : 1] : \zeta \in \mathbb{C}\} \cup \{[1 : 0]\}, \tag{11}$$

and we can see by comparison with Eq. (10) that the set on the left is isomorphic to $\mathbb{C}$ while $[1 : 0]$ is equivalent to the point at infinity, thus demonstrating its correspondence to $\overline{\mathbb{C}}$ as constructed above. Notice what we have done here is construct the Bloch sphere in one step: when we constructed it in Section III, we normalised our $\mathbb{C}^2$ vectors first (quotienting out the real modulus), and afterwards quotiented out the global phases $e^{i\theta}$, meaning that in total we quotiened out multiplication by a complex number in modulus-argument form. This is exactly what we have done here, demonstrating the equivalence of $\mathbb{CP}^1$ and the Bloch sphere. Since we have also shown equivalence to the Riemann sphere, which is topologically equivalent to S^2, we have shown that $S^2 \cong \mathbb{CP}^1$ as stated in Section II.

B. Spinors according to Riemann (sphere)

Now, equipped with the Riemann sphere and its connection with $\mathbb{CP}^1$ and the Bloch sphere, we are ready to discuss how to represent rotation axis and direction using spinors. Let

$$\chi \in \mathbb{C}^2, \quad \chi = [\chi_1, \chi_2] \tag{12}$$

denote what will become a spinor, and since this will represent a quantum state, assume it is normalised such that $|\chi_1|^2 + |\chi_2|^2 = 1$. (Normally we would use column vectors here, but we'll stick to row vectors for now.) For a vector $(x, y, z) \in S^2$, we can use Eq. (8) to get the corresponding point on the Riemann sphere:

$$(x, y, z) \mapsto \frac{x + iy}{1 + z} = \zeta, \tag{13}$$

which we can suggestively write as the ratio of two complex numbers

$$\zeta = \frac{\chi_1}{\chi_2}, \quad \chi_1 = \alpha(x + iy), \quad \chi_2 = \alpha(1 + z). \tag{14}$$

where $\alpha \in \mathbb{C}$ is a normalisation constant. It turns out that

$$[\chi_1, \chi_2] = [\alpha(x + iy), \alpha(1 + z)] \tag{15}$$

is a spinor representation of (x, y, z) [2], and we can use the normalisation requirement to show that

$$|\alpha| = \frac{1}{\sqrt{2(1 + z)}}. \tag{16}$$

However, $\theta := \arg \alpha$ is a free parameter in this representation, so there is an infinite family of spinors that represent this S^2 vector, given by the general spinor representation

$$\chi = \left[\frac{x + iy}{\sqrt{2(1 + z)}} \mathrm{e}^{i\theta}, \sqrt{\frac{1 + z}{2}} \mathrm{e}^{i\theta} \right] .I \tag{17}$$

This form can be immediately connected to the Bloch sphere, as the immeasurable $e^{i\theta}$ global phase has been extracted from the rest of the expression. Thus we can quotient it out to get the physical state

$$\chi \sim \left[\frac{x + iy}{\sqrt{2(1 + z)}}, \sqrt{\frac{1 + z}{2}} \right]. \tag{18}$$

This is a basic realisation of the map $S^3 \to S^2$ referenced in Section II, called the Hopf fibration, which we will discuss more abstractly (and more generally) in Section VI.

Now, if we are given a spinor $[\chi_1, \chi_2]$ with $\chi_2 \neq 0$, we can always divide by χ_2 to get

$$\chi \sim \left[\frac{x + iy}{1 + z}, 1 \right] \in \left[\frac{x + iy}{1 + z} : 1 \right] \sim \frac{x + iy}{1 + z} \in \overline{\mathbb{C}}, \tag{19}$$

so such a spinor can always be associated with a unique equivalence class $[\zeta : 1] \in \mathbb{CP}^1$, and hence with a unique complex number ζ on the Riemann sphere. Similarly, when $\chi_2 = 0$, the corresponding equivalence class is $[1 : 0]$, and hence the corresponding point on the Riemann sphere is ∞. This is the mathematical realisation of the equivalence of the Bloch sphere, Riemann sphere, and $\mathbb{CP}^1$.

So up to global phase, spinors are just equivalent to numbers on the Riemann sphere! This seems to be making things more complicated for no reason, but as we'll see here and in Part 2, it's a very powerful change in perspective, as the properties of spinors we listed at the beginning of Section V become much easier to show and visualise.

C. Spinors, Orthogonality, and Antipodal States

First, while every vector $(x, y, z) \in S^2$ has a U(1) family of non-identical spinors representing it, a given spinor always uniquely determines the vector (x, y, z) it represents, which can be found by first converting it to the equivalent complex number and then using the inverse stereographic projection.

Second, antipodal vectors in S^2 have orthogonal spinor representations, and vice versa. The forward implication is just a simple calculation: use Eq. (8) to get the spinor representations of (x, y, z) and $(-x, -y, -z)$ and take their inner product, simplifying using the fact that $x^2+y^2+z^2 = 1$ to show that it is zero. In fact, we can replace the spinor representations of (x, y, z) and $(-x, -y, -z)$ with any elements of the corresponding $\mathbb{CP}^1$ equivalence classes and still get an inner product of 0, confirming that the spinor representations for opposite physical spin states never overlap, regardless of phase. The reverse implication requires showing orthogonal spinors always represent antipodal vectors, and this can be shown using a combination of algebraic and geometric arguments. Suppose $[a, b]$ and $[c, d]$ are orthogonal spinors, and further suppose we have eliminated the global phase so that b and d are both real. Then (recalling $*$ denotes complex conjugation), orthogonality gives us

$$ac^* + bd = 0 \iff \frac{|a|}{b} e^{i \arg(a)} = \frac{d}{|c|} e^{i(\arg(c)-\pi)}, \tag{20}$$

which can only be satisfied without breaking the normalisation of the spinors if

$$d = |a|, |c| = b, \arg(c) = \arg(a) + \pi. \tag{21}$$

Thus we must have

$$[c, d] = \left[b e^{i(\arg(a)+\pi)}, |a|\right], \tag{22}$$

but exploiting the fact we noticed above regarding orthogonality being independent of the choice of representatives, we can divide by $e^{i(\arg(a)+\pi)}$ to choose the more interpretable representative

$$[c, d] \sim \left[b, |a| e^{i(\arg(a)+\pi)}\right] = [b^*, -a^*], \tag{23}$$

where we have used that b is real so that $b = b^*$. Clearly, the inner product of $[a, b]$ and $[b^*, -a^*]$ is still zero as we expect, but now the corresponding Riemann sphere equivalents of the physical states are

$$[a, b] \sim \left[\frac{a}{b} : 1\right] \sim \frac{a}{b} \in \overline{\mathbb{C}} \tag{24}$$

and

$$[b^*, -a^*] \sim \left[-\frac{b^*}{a^*} : 1\right] \sim -\frac{b^*}{a^*} = -\left(\left(\frac{a}{b}\right)^{-1}\right)^* \in \overline{\mathbb{C}}. \tag{25}$$

This latter number is just $\frac{a}{b}$ after undergoing inversion, complex conjugation, and multiplication by -1, the composition of which can be easily interpreted geometrically in polar coordinates:

$$\frac{a}{b} = re^{i\theta} \mapsto \frac{1}{r} e^{-i\theta} \mapsto \frac{1}{r} e^{i\theta} \mapsto \frac{1}{r} e^{i(\theta+\pi)} \tag{26}$$

So this process rotates $\frac{a}{b}$ a half turn about 0 and inverts its modulus, which when viewed through the lens of the inverse stereographic projection is exactly the process that sends a number on the Riemann sphere to its antipodal point! More loosely, since the quotienting process makes points separated by a half turn about the origin (like $[x, y]$ and $[-x, -y]$) equivalent, half turns in $\mathbb{C}^2$ become full turns in $\mathbb{CP}^1$, and quarter turns become half turns. Thus the seemingly strange tendency for orthogonal spinors to correspond to antipodal states makes perfect sense.

VI. FIBER BUNDLES

Equipped with our mathematical understanding of spinors and orthogonality, we are now in a position to understand the most general construction of the mapping from spinors to physical states, using fiber bundles and the Hopf fibration.

Let E, B, and F be topological spaces. E is our *total space*, which can be thought of as the big space that we want to map to the "smaller" *base space* of the bundle B. In some small neighborhood of our total space, it is said to "look like" the *product space* $B \times F$; more precisely, there is a homeomorphism φ from $V \subset E$ to a local neighborhood $W \subset B \times F$. This is called *local triviality.* The total space does not necessarily have the same global topology as the product space, which is what makes the mapping π (called the projection of the bundle) from the total space to the base space unique. Furthermore, there exists a map proj_1, called projection to the first factor, that takes us from the product space $B \times F$ to the base space B, forming the commutative diagram:

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r, "\varphi"] \arrow[d, "\pi"'] & U \times F \arrow[dl, "\mathrm{proj}_1"] \\ U & \end{tikzcd}$$

where $U \subset B$ is a subset of the base space, $U \times F$ is a subset of the product space, and $\pi^{-1}(U) \subset E$ is a subset of the total space. The specific case where the total space and product space coincide is called the trivial fiber bundle, and the projection of the fiber π is just projection onto the first factor.

Hopf Fibration

We finally reach the meat and potatoes of the matter. This is what all the set up and construction has been working towards for the desired Bloch sphere visualization. We pick up the story here with spinors S^3 in $\mathbb{C}^2$.

A necessary piece of the puzzle that takes us from S^3 to S^2 is a theorem stating that if G is a Lie Group and H is a closed Lie subgroup, there exists a projection π to the quotient G/H as a fiber bundle $(G, G/H, \pi, H)$ [3]. This is how we will go about defining the Hopf fibration, which is the proper name for the map $S^3 \to S^2$ that we have

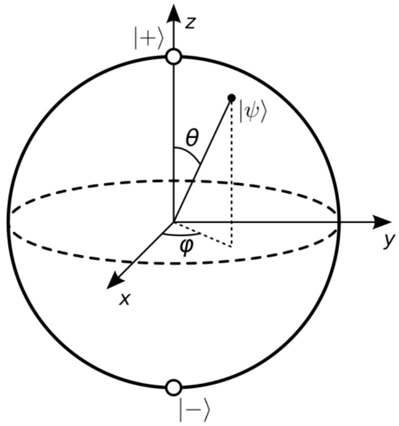

FIG. 2. Bloch sphere with (orthogonal) spin up and spin down states marked at antipodal points. [4]

been using to move from spinors to physical states all this time. Our Lie group G is SU(2). Each element can be written as

$$U = \begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix}, \tag{27}$$

where the $\det U = 1$ condition translates to $|\alpha|^2 + |\beta|^2 = 1$. Thus by our previous discussion of orthogonal spinors, this is equivalent to writing

$$U = \big(u, v\big), \tag{28}$$

where

$$u = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}, v = \begin{bmatrix} -\beta \\ \alpha \end{bmatrix}$$

are orthogonal spinors. Hence SU(2) matrices are one-to-one with spinors (since (u, v) and (v, u) are distinct and can be associated with u and v respectively), confirming that SU(2) is isomorphic to S^3. But since we can quotient global phases out of spinors to get physical states, maybe we can do that here too! Choosing our closed Lie subgroup H to be the circle group U(1), careful application of Lie theory shows us that $(\mathrm{SU}(2), \mathrm{SU}(2)/\mathrm{U}(1), \pi, \mathrm{U}(1))$ is in fact a fiber bundle, where our projection π is a quotient map from our spinors in $S^3 \cong \mathrm{SU}(2)$ to SU(2)/U(1) such that $|\psi\rangle \sim e^{i\theta}|\psi\rangle$. This means that SU(2) matrices aren't just transformations of spinors, they can represent spinors themselves! So rather than having separate states and transformations, we can just use SU(2) matrices to represent everything.

Thus all our previous results for spinors can now be applied to SU(2) matrices. In particular, quotienting by U(1) sends matrix U in Eq. (27) through the chain

$$\begin{pmatrix} \alpha & -\beta^* \\ \beta & \alpha^* \end{pmatrix} \mapsto [\alpha, \beta] \mapsto \begin{cases} \left[\frac{\alpha}{\beta} : 1\right] \mapsto \frac{\alpha}{\beta} & \text{if } \beta \neq 0 \\ [1:0] \mapsto \infty & \text{otherwise} \end{cases} \tag{29}$$

meaning we can associate each SU(2) matrix with an equivalence class in $\mathbb{CP}^1$ (i.e. a physical state) and a point on the Riemann sphere (i.e. a point on S^2). Thus the quotient SU(2)/U(1) is isomorphic to S^2, and since $S^3 \cong \mathrm{SU}(2)$, this means the fiber bundle projection

$$\pi : \mathrm{SU}(2) \to \mathrm{SU}(2)/\mathrm{U}(1) \tag{30}$$

is also a map

$$\pi : S^3 \to S^2, \tag{31}$$

and this is the general definition of the Hopf fibration. What makes all the setup we've done necessary is that the total space S^3 is not trivially $S^2 \times S^1$ (where we simply take the Cartesian product of the fiber U(1) with the Bloch sphere). Globally their topologies are different, so care needs to be taken when constructing the projection mapping.

Thus we have finally come full circle, having arrived at the most general definition of the Bloch sphere (i.e. SU(2)/U(1)) for a spin-$\frac{1}{2}$ system, and having properly identified how the Hopf fibration is fundamentally responsible for our ability to associate spinors with physical states in a consistent way. Next time, we'll see much more of SU(2), properly analysing why it is the main group that transforms spinors and showing how it arises naturally when we identify rotations of S^2 with rotations of the Riemann sphere. Featuring examples that can be interactively plotted by readers at home, we will reveal the underlying structure connecting rotations to spinors, and finally explain why rotating physical states by a given angle only rotates the spinor by half the angle.

[1] G. A. Jones and D. Singerman, *Complex functions: an algebraic and geometric viewpoint* (Cambridge university press, 1987).

[2] G. Fano and S. Blinder, *Twenty-first century quantum mechanics: Hilbert space to quantum computers* (Springer, 2017).

[3] J. M. Lee, *Introduction to Smooth Manifolds*, Graduate Texts in Mathematics, Vol. 218 (Springer, 2012).

[4] Smite-Meister at Wikimedia Commons, Published under the Creative Commons Attribution-Share Alike 3.0 Unported license (https://creativecommons.org/licenses/by-sa/3.0/deed.en), modified so the spin up and down states are labelled $|+\rangle$ and $|-\rangle$ instead of $|1\rangle$ and $|0\rangle$ respectively.