Quantum Probability Measured

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Exploring the pure mathematics background that underscores probability in quantum mechanics.

I. INTRODUCTION

Quantum systems present themselves as some of the most mathematically interesting systems to be studied. Over the course of history, their mathematical descriptions have been fleshed out, allowing us to reap the benefits with simplified and elegant formalisms for calculating probabilities and expectation.

Measure theory is a field of mathematics that plays an important role in physics. It provides the language to rigorously define the physical ideas of mass, momentum, and probability in mathematical language. To illuminate intuition behind some of the familiar equations one comes across in undergraduate physics, we will explore measure theory's underpinning in quantum mechanics and look at a specific measure (the Projection-Valued Measure), all in a short-form style.

II. MEASURE THEORY IN 5 MINUTES

In this section, we will see the motivation from measure theory that underlies the probability theory used. For physics students who have not done measure theory, I implore you to go through the properties of measures.

Measure? σ-algebra?

Measure theory is a branch of mathematics that generalizes notions of distance, length, area, and probability by abstracting integration. I will speed through some of the fundamentals of measure theory to provide some intuition and introduce the concepts that inform our descriptions of quantum systems.

Before we discuss what a "measure" is, let us discuss the domains on which they operate: "σ-algebras". We let $\mathcal{A}$ be a σ-algebra, defined over subsets of some set X. The σ-algebra consists of subsets of X, along with their respective set complements, the empty set, and must be closed under countable union.[1]

A *measure* is a set function $\mu : \mathcal{A} \to \mathbb{K}$, where $\mathbb{K}$ is a codomain like the extended reals ($\bar{\mathbb{R}}$). Briefly, measure μ is monotonically increasing, $\mu(\emptyset) = 0$, and respects 'countable additivity'.[1]

Putting the previous concepts together, we define the measure space as $(X, \mathcal{A}, \mu)$. Different measure spaces utilize different measures and σ-algebras. Depending on our given system, we can change whether our measure maps to a subset of the extended real line, or some other codomain entirely. Using our measure spaces, we want to generalize integration, like making up for where the classical Riemann integral fails. 'Measurable functions' can be integrated to achieve this purpose. An application of this is in probability theory.

Probability Theory

One important instance of a measure space is a 'probability space'. We can have a probability space $(\Omega, \mathcal{A}, \mathbb{P})$, where Ω is our sample space and $\mathcal{A}$ is a σ-field (same idea as σ-algebra) containing outcomes of events. The measure on this space $\mathbb{P} : \mathcal{A} \to [0, 1]$ is our probability measure. The probability space has the property $\mathbb{P}(\Omega) = 1$.

In the context of probability theory, measurable functions are called 'random variables'. These functions are important when defining measures with a density, where a measure μ equals $g(x)\nu$- where g is a measurable function and ν is a measure.[2]

Abstracting our integrals allows us to integrate functions over sets with respect to a measure. Thus, we can integrate with respect to our probability measure to find the expected value of a random variable:

$$\mathbb{E}[X] = \int_{\mathbb{R}} x \, d\mathbb{P}_X$$

using what we know about measures with a density, we can alter the above integral to take the form:

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x) dx$$

where f is our probability density function and the integral in terms of the Lebesgue measure.[2]

We will find that the abstract integral form of the expected value is reflected when discussing our quantum systems using the projection-valued measure.

III. PROJECTION-VALUED MEASURE

Okay, the meat and potatoes of the matter. Whatever you want to call it, this is the main conceptual chunk. Given some idea of what measure theory is and how it informs probability theory, let us look at the axioms for the projection-valued measure, then see how we derive expressions for expectation and probability in quantum mechanics.

Definition 1

The projection-valued measure (PVM) [3] on $\mathbb{R}$ is a map $P : \mathcal{B}(\mathbb{R}) \to \mathcal{L}(\mathcal{H})$ from the Borel σ-algebra, $\mathcal{B}(\mathbb{R})$, to the space of bounded linear operators on the Hilbert space $\mathcal{H}$ with properties:

1. $\forall B \in \mathcal{B}(\mathbb{R})$, $P(B)$ is an orthogonal projection: $P(B) = P(B)^2$ and self-adjoint $P(B) = P(B)^*$.

2. $P(\emptyset) = 0$, $P(\mathbb{R}) = I$, where I is the identity operator.

3. For $B_i, B_j \in \mathcal{B}(\mathbb{R})$ where $B_i \cap B_j = \emptyset$ for $i \neq j$,
$$P\left(\bigcup_{n\in\mathbb{N}} B_n\right) = \sum_{n=1}^{\infty} P(B_n)$$

Similar to the previous concept of a real-valued measure that returns scalars in $\mathbb{R}$, the PVM measures sets in the space and returns self-adjoint projections associated with that observable's operator, say $\hat{A}$ in this case.[4] Specifically, if the Borel σ-algebra corresponds to measurable outcomes for a given observable, then for some Borel set $B \in \mathcal{B}(\mathbb{R})$, $P(B)$ corresponds to an operator whose eigenvalues are a subset of the measurable outcomes of the original observable $\hat{A}$. This means that P maps $|\psi\rangle$ to subspaces of $\mathcal{H}$.[4]

For example, take a simple discrete binary system: spin up and down. The eigenvalues for the spin operator associated with this observable are $\{\pm 1\}$. We can get the operators

$$P(\{+1\}) = |+\rangle\langle+|$$
$$P(\{-1\}) = |-\rangle\langle-|$$

where $|-\rangle$ and $|+\rangle$ span the state space.

Therefore, if we take some arbitrary state vector $|\psi\rangle$, then the projection-valued measure acting on this vector maps it to a subspace of the Hilbert space:

$$\begin{aligned} P(\{+1\})|\psi\rangle =& (|+\rangle\langle+|)(\alpha|+\rangle + \beta|-\rangle) \\ =& \alpha|+\rangle \end{aligned}$$

where $\alpha, \beta \in \mathbb{C}$. (Note this new state isn't inherently normalized). Similar argument for $P(\{-1\})$.

PVM's in Probability

We also get the form for the probability for observing a given outcome $B \in \mathbb{R}$ (a range of eigenvalues):

$$\begin{aligned} \text{Probability}(B) &= \| P(B)|\psi\rangle \|^2 \\ &= \langle\psi| P(B)^2 |\psi\rangle \\ &= \langle\psi| P(B) |\psi\rangle \\ &= \langle\varphi|\psi\rangle . \end{aligned}$$

$|\varphi\rangle$ is the projection of $|\psi\rangle$ to a subspace of $\mathcal{H}$.

Expectation value is defined as the weighted sum:

$$\begin{aligned} \mathbb{E}(\hat{A}) &= \sum_{n=0}^{\infty} \lambda_n \langle\psi| P_n |\psi\rangle \\ &= \langle\psi| (\sum_{n=0}^{\infty} \lambda_n P_n) |\psi\rangle \\ &= \langle\psi| \hat{A} |\psi\rangle \end{aligned}$$

The above derivation utilizes the spectral theorem in the third line. We can generalize the above result to a continuous spectrum:

$$\begin{aligned}\langle\psi|\,\hat{A}\,|\psi\rangle &= \int_{\mathbb{R}} \lambda\,\langle\psi|\,dP(\lambda)\,|\psi\rangle \\ &= \int_{\mathbb{R}} \lambda\, d(\langle\psi|\,P(\lambda)\,|\psi\rangle)\end{aligned}$$

Here is the key, we can treat $\langle\psi|\,P(\lambda)\,|\psi\rangle$ as a probability measure as it adheres to the rules of a probability measure.[5] This allows us to show equivalence between the expectation value above and the expected value from the probability theory section, as they both relate to integrating a variable with respect to a probability measure.[6]

Here is quick reminder of the Spectral Theorem from the previous derivation. It states that any Hermitian operator $\hat{H}$ can be expressed as

$$\hat{H} = \int_{\mathrm{spec}(\hat{H})} \lambda dP(\lambda)$$

where λ are the observable's eigenvalues (a generalization of the set of eigenvalues), $\mathrm{spec}(\hat{H})$ is the spectrum of $\hat{H}$, and $dP(\lambda)$ is the projection-valued measure of the observable.[6]

BUT WAIT, THERE'S MORE!

PVM's have plenty of applications in proofs in quantum mechanics beyond what I have covered in this article. We have seen this measure used in simple concepts of probability, but it has tie-ins into advanced topics like systems of quantum-identical particles.

PVMs have a further generalization that being Positive Operator-Valued Measures (POVMs). POVMs are measures whose values are positive semi-definite operators on a Hilbert space. They can be used in advanced proofs in quantum field theory or quantum information theory. They also have applications in simpler systems. For instance, if you wanted to reconstruct an unknown state by performing measurements, a process known as state tomography, you can employ POVMs to improve efficiency.[7]

The story of how measures inform quantum theory is a deep and rich area of mathematics and physics. I hope to have helped bring forth some of the interesting ideas for your consideration and as a nice form of small talk next time you're at the water cooler.

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